Efficient SAT-based Minimal Model Generation Methods for Modal Logic S5

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Syntax

- Language $\mathcal{L}$ extends the propositional language with the modal connectives $\Box$ and $\Diamond$.

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box\phi \mid \Diamond\phi$$

where $p \in \mathbb{P}$ and $\mathbb{P}$ denotes a countably infinite non-empty set of propositional variables.

Semantics

- Standard Kripke semantics for modal logic defines a frame, which consists of a non-empty set W of **possible worlds**, and a **binary relation** R.

$$W \text{ — Set of possible worlds}$$
$$R \text{ — Accessibility relation}$$

- Function I: $W \times R \to \{0, 1\}$.

Axiom

- These axioms imply that the relation R is **reflexive**, **symmetric** and **transitive**.

$$\mathcal{K}.\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$
$$\mathcal{T}.\Box A \rightarrow A$$
$$\mathcal{B}.A \rightarrow \Box\Diamond A$$
$$4.\Box A \rightarrow \Box\Box A$$

Satisfiability Relation

- The satisfiability relation $\models$ for formulas in $\mathcal{L}$ is recursively defined as follows:

$$(W, I, w) \models \top$$
$$(W, I, w) \models p \text{ iff } I(w, p) = 1$$
$$(W, I, w) \models \neg\phi \text{ iff } (W, I, w) \not\models \phi$$
$$(W, I, w) \models \phi \wedge \varphi \text{ iff } (W, I, w) \models \phi \text{ and } (W, I, w) \models \varphi$$
$$(W, I, w) \models \phi \vee \varphi \text{ iff } (W, I, w) \models \phi \text{ or } (W, I, w) \models \varphi$$
$$(W, I, w) \models \Box\phi \text{ iff } \forall w' \in W, (W, I, w') \models \phi$$
$$(W, I, w) \models \Diamond\phi \text{ iff } \exists w' \in W, (W, I, w') \models \phi$$

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Applications

- Knowledge Compilation

Niveau, A., & Zanuttini, B. (2016, July). Efficient representations for the modal logic S5. In Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI'16.

Bienvenu, M., Fargier, H., & Marquis, P. (2010, July). Knowledge compilation in the modal logic S5. In Proceedings of the AAAI Conference on Artificial Intelligence AAAI'10.

- Epistemic planner

Wan, H., Yang, R., Fang, L., Liu, Y., & Xu, H. (2015, June). A complete epistemic planner without the epistemic closed world assumption. In Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI'15.

Problems

1. **[S5-Satisfiability (S5-SAT)]** Determining if there exists a model (W, I, w) that satisfies a given S5 formula θ.
2. **[S5-K-Satisfiability (S5-K-SAT)]** Determining if there exists a model (W, I, w) where $|W| = K$ that satisfies a given S5 formula θ.
3. **[Minimal S5-Satisfiability (MinS5-SAT)]** Finding a model (W, I, w) that satisfies a given S5 formula θ and it has no model (W', I', w') such that $|W'| < |W|$.

S5-NF

- S5-NF is a kind of CNF-like first degree normal form which is made up of **S5-literals**:

Definition (S5-literal)

Propositional literal: p e.g.$\neg q$, q
B-literal: $\Box(p \vee q \vee \ldots \vee r)$ e.g. $\Box(\neg p \vee q \vee \neg r)$
D-literal: $\Diamond(p \wedge q \wedge \ldots \wedge r)$ e.g. $\Diamond(\neg p \wedge \neg r)$

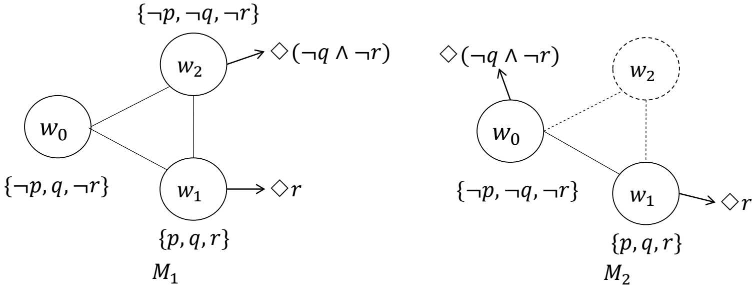
Example (An S5 formula θ and its S5-NF ϕ)

$$\theta = \Diamond\Box((r \to p \wedge q) \wedge (\Diamond(\neg r \to \neg p \wedge q))) \wedge (\neg p \to \neg\Box(q \wedge r))$$

$$\phi = \underbrace{\Box(p \vee q \vee \neg r)}_{C_1} \wedge \underbrace{\{\Diamond(\neg p \wedge q) \vee \Diamond r\}}_{C_2} \wedge \underbrace{\{p \vee \Diamond(\neg q \wedge \neg r)\}}_{C_3}$$

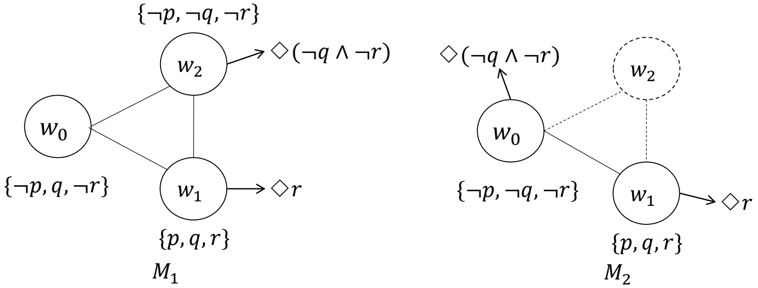
Model

- $\phi = \underbrace{\Box(p \vee q \vee \neg r)}_{C_1} \wedge \underbrace{\{\Diamond(\neg p \wedge q) \vee \Diamond r\}}_{C_2} \wedge \underbrace{\{p \vee \Diamond(\neg q \wedge \neg r)\}}_{C_3}$



Model

- The key is how to assign the possible worlds to D-literals.



Two Paradigms

1. Querying SAT Iteratively
2. Solving via MaxSAT

Via SAT

- Propositional variable p_j to denote the truth value of p in the possible world w_j. Translation function $tr^{-}_{SAT}(\phi, K)$ can produce a propositional formula for an input S5-NF ϕ with K possible worlds:

1. $\top \Rightarrow \top \quad \bot \Rightarrow \bot$
2. For all propositional literals p in ϕ: $p \Rightarrow p_0$
3. For all B-literals in ϕ:

$$\Box(p \vee q \vee \cdots \vee s) \Rightarrow \bigwedge_{j=0}^{K-1} (p_j \vee q_j \vee \cdots \vee s_j)$$

4. For all D-literals in ϕ:

$$\Diamond(p \wedge q \wedge \cdots \wedge r) \Rightarrow \bigvee_{j=0}^{K-1} (p_j \wedge q_j \wedge \cdots \wedge r_j)$$

Via SAT

- The MinS5-SAT problem can be seen as an optimization problem:

$$\textbf{minimize} \quad K$$
$$\textbf{s.t.} \quad tr^{-}_{SAT}(\phi, K) \text{ is satisfiable.}$$

Via SAT

- If an S5-NF has m clauses with diamond operators, then the upper bound of the number of possible worlds $\mu \leq m$.
  1. $K = 1 \sim \mu$
  2. $K = \mu \sim 1$
  3. Binary search for the minimal K

Via MaxSAT

- For each possible world w_j, add a switch Boolean variable v_j to open or close it.
  1. When v_j is false, the possible world w_j will be closed.
  2. When v_j is true, the possible world w_j will be open.

Via MaxSAT

- Translation function $tr^{-}_{PMS}(\phi, \mu)$ can produce a partial MaxSAT formula for an input S5-NF ϕ with at most μ (upper bound) possible worlds:
  1. $\top \Rightarrow \top \quad \bot \Rightarrow \bot$
  2. For all propositional literals: $p \Rightarrow p_0$
  3. For all B-literals:
     $$\Box(p \vee q \vee \cdots \vee s) \Rightarrow \bigwedge_{j=0}^{\mu-1} (p_j \vee q_j \vee \cdots \vee s_j)$$
  4. For all D-literals:
     $$\Diamond(p \wedge q \wedge \cdots \wedge r) \Rightarrow \bigvee_{j=0}^{\mu-1} (v_j \wedge p_j \wedge q_j \wedge \cdots \wedge r_j)$$
  5. Add a unit clause: v_0
  6. Add unit soft clauses: $\bigwedge_{j=1}^{\mu-1} (\neg v_j)$

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Via MaxSAT

$$\phi = \underbrace{\Box(p \vee q \vee \neg r)}_{C_1} \wedge \underbrace{\{\Diamond(\neg p \wedge q) \vee \Diamond r\}}_{C_2} \wedge \underbrace{\{p \vee \Diamond(\neg q \wedge \neg r)\}}_{C_3}$$

Example

If the upper bound μ for the S5-NF ϕ in is 3, then the $tr^{-}_{PMS}(\phi, 3)$ is :

$$\bigwedge_{j=0}^{2}(p_j \vee q_j \vee \neg r_j) \wedge \{\bigvee_{j=0}^{2}(v_j \wedge \neg p_j \wedge q_j) \vee \bigvee_{j=0}^{2}(v_j \wedge r_j)\}$$

$$\wedge\{p_0 \vee \bigvee_{j=0}^{2}(v_j \wedge \neg q_j \wedge \neg r_j)\} \wedge v_0 \wedge \underbrace{\neg v_1}_{Soft} \wedge \underbrace{\neg v_2}_{Soft}$$

SIF Strategy

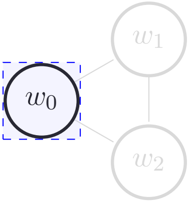
- **Smallest Index First** a static symmetry breaking technique

Attention

1. The key to find a minimal model is searching for an optimal assignment of possible worlds for satisfied D-literals.
2. Each D-literal only need one world.

SIF Strategy

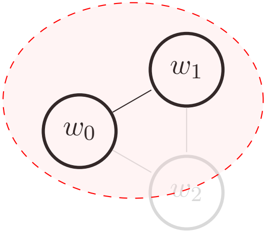
- Considering C_1.



$$\phi = \underbrace{\Box(p \vee q \vee \neg r)}_{C_1} \wedge \underbrace{\{\Diamond(\neg p \wedge q) \vee \Diamond r\}}_{C_2} \wedge \underbrace{\{p \vee \Diamond(\neg q \wedge \neg r)\}}_{C_3}$$

SIF Strategy

- Considering C_1.
- Considering C_2.



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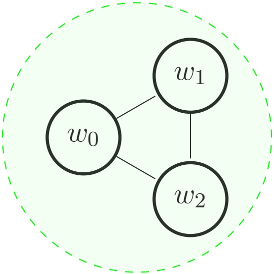
SIF Strategy

- Considering C_1.
- Considering C_2.
- Considering C_3.



$$\phi = \underbrace{\Box(p \vee q \vee \neg r)}_{C_1} \wedge \underbrace{\{\Diamond(\neg p \wedge q) \vee \Diamond r\}}_{C_2} \wedge \underbrace{\{p \vee \Diamond(\neg q \wedge \neg r)\}}_{C_3}$$

SIF Strategy

- $\Omega_1 = \{l \in \mathbb{N} | 0 \leq l \leq L-1\}$ and $\Omega_2 = \{l \in \mathbb{N} | L \leq l \leq K-1\}$. Initially $L = 1$, $\Omega_1 = \{0\}$, $\Omega_2 = \{1, ..., K-1\}$.
- Whenever the translation procedure encounters an S5-clause which has D-lierals, update L to $Min(L+1, K-1)$.

SIF Strategy

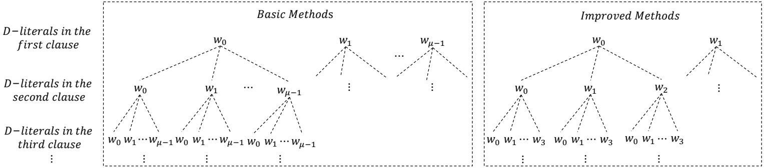
- Translation function $tr_{SAT}(\phi, K)$ is the improved version of $tr^{-}_{SAT}(\phi, K)$ with the SIF strategy:
  When the procedure is translating S5-clause C_i, update L, iff C_i has D-literals. For all D-literals in C_i:
  $\Diamond(p \wedge q \wedge \cdots \wedge r) \Rightarrow \bigvee_{j \in \Omega_1} (p_j \wedge q_j \wedge \cdots \wedge r_j)$

SIF Strategy

- Translation function $tr_{PMS}(\phi, \mu)$ is the improved version of $tr^{-}_{PMS}(\phi, \mu)$ with the SIF strategy:
  (i) Add : $\bigwedge_{j=0}^{\mu-2} (v_{j+1} \rightarrow v_j)$
  When the procedure is translating S5-clause C_i, update L, iff C_i has D-literals.
  (ii) For all D-literals in C_i:
  $\Diamond(p \wedge q \wedge \cdots \wedge r) \Rightarrow \bigvee_{j \in \Omega_1} (v_j \wedge p_j \wedge q_j \wedge \cdots \wedge r_j)$

SIF Strategy

- The Benefit of SIF



The search spaces of the basic and improved methods.

Experimental Evaluation

Table: The comparison of efficiency on all benchmarks. "-" means no instance can be solved within the time bound(300s).

Ins (#Total)	Inc.			Dec.			Bs.			PMS.			S52SAT1.0		Lck.	
	$\#Win$	T_{avg}	Mem	$\#Win$	T_{avg}	Mem	$\#Win$	T_{avg}	Mem	$\#Win$	T_{avg}	Mem	T_{avg}	Mem	T_{avg}	Mem
$QMLTP$(41)	41	0.5	0.8M	0	1.3	1.0M	0	0.91	0.9M	0	1.1	20.1M	78.90	25.0M	43910.9	21G
$QS5\_1$(252)	248	1995.6	4.8G	0	29745.3	5.5G	0	6800.64	5.3G	4	5846.4	6.6G	107378.2	64G	-	-
$QS5\_2$(240)	149	129303.9	3.1G	0	189785.7	8.5G	0	164857.9	6.2G	5	181272.8	8.8G	290205.8	64G	-	-
$3CNF$ (55)	54	19476.2	62.8M	0	223919.6	109.7M	0	40669.2	76.1M	0	238946.2	250.1M	126804.3	2.35G	-	-
$3CNF_{mu}$(945)	939	7776.2	59.9M	0	207520.2	100.7M	0	48329.9	55.5M	0	187422.1	174.0M	290205.9	64G	-	-
LWB_k (42)	42	154.5	46.8M	0	180.6	51.5M	0	166.7	48.8M	0	176.4	59.4M	81326.9	64G	147467.9	64G
LWB_kt (105)	105	7.8	48.1M	0	15.6	55.1M	0	14.2	55.9M	0	18.4	62.8M	30916.2	64G	61103.1	58G
LWB_s4(105)	105	40.4	6.8M	0	47.6	7.1M	0	46.5	7.1M	0	51.8	7.7M	38299.4	24.5G	57393.4	62G
$qbfS$ (177)	177	1.59	3.0M	0	5.78	4.8M	0	3.19	3.0M	0	3.24	12.4M	591.44	84.0M	-	-

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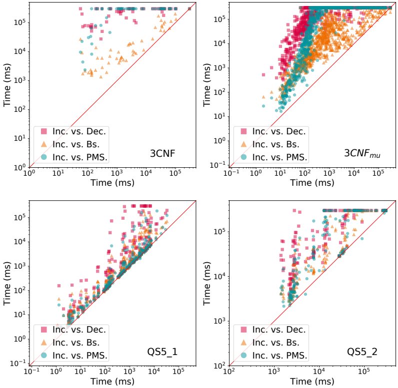
S5cheetah 2.0 and benchmarks: http://www.square16.org/tools/s5cheetah/

Experimental Evaluation



The comparison of running time on $3CNF$ and $QS5$.(The x-axis corresponds to the time used by incremental method and the y-axis corresponds to the time used by other methods.)

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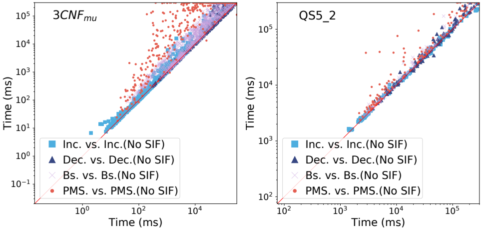
Experimental Evaluation



The comparison of running time on $3CNF_mu$ and $QS5\_2$. (The x-axis corresponds to the time used by methods with SIF and the y-axis corresponds to the time used by methods without SIF.)

Upper bounds

- $m+1$, m is number of modal operators. (1977)
- $dd(\theta)+1$, $dd(\cdot)$ is diamond degree.(AAAI-2017)
- $\chi+1$, χ is reasoned from S5-NF.(IJCAI-2019)

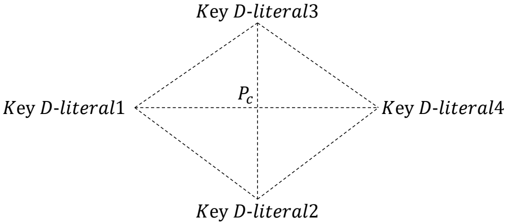
Experimental Evaluation

Table: The minimal number of possible worlds VS. estimated upper bounds.

Ins	MinW	$\chi + 1$	$dd(\theta) + 1$	$m + 1$
$QMLTP$	1.31	1.75	6.87	242.73
$QS5\_1$	1.00	16.05	2826.70	8333.36
$QS5\_2$	1.92	40.93	1329.41	35095.44
$3CNF$	7.03	125.45	152.82	427.85
$3CNF_{mu}$	6.89	233.47	316.50	797.16
LWB_k	1.50	3.47	440.92	1049.48
LWB_kt	1.40	4.00	217.89	1035.52
LWB_s4	1.20	2.57	236.22	1130.41
$qbfS$	2.00	3.00	45.07	1637.87

Analyses

- Why is a minimal model small?
  - Key D-literals — D-literals that have to be true to satisfy ϕ.
  - Conflict edges — Two D-literals can't be true in the same world.



The conflict graph of four key D-literals.

Analyses

- The chromatic number

$$\chi(G) \leq \delta = \max_{i \in V} min(d_i + 1, i)$$

where d_i is the degree of vertex i and $d_1 \geq d_2 \geq ... \geq d_{|V|}$.
Then the probability that $\chi(G)$ less than constant H is:

$$P(\delta \leq H) = P(|\{d_i | d_i \geq H\}| \leq H)$$

Analyses

- Independent random variable $X_{ij}(i, j \in \{1, 2, ..., m\})$ denotes the appearance of edge between two vertexes i and j.Based on central limit theorem, we have:

$$Z = \frac{\sum_{j=1}^{m} X_{ij} - mp_k^2 p_c}{\sqrt{mp_k^2 p_c(1 - p_k^2 p_c)}} \sim N(0, 1)$$

So, $P(d_i \geq H) = \Phi(1 - \frac{H - mp_k^2 p_c}{\sqrt{mp_k^2 p_c(1 - p_k^2 p_c)}})$ where Φ is the cumulative distribution function (CDF) of normal distribution $N(0, 1)$.

Analyses

- We simply use independent random variable Y_i to denote whether $d_i \geq H$ and $P(Y_i = 1) = P(d_i \geq H) = p_d$. Then we have:

$$P(|\{d_i|d_i \geq H\}| \leq H) = P(\sum_{i=1}^{m} Y_i \leq H)$$
$$= P(\frac{\sum_{i=1}^{m} Y_i - mp_d}{\sqrt{p_d(1-p_d)}} \leq \frac{H - mp_d}{\sqrt{p_d(1-p_d)}})$$

Reuse the central limit theorem, we have:

$$P(\delta \leq H) = P(|\{d_i|d_i \geq H\}| \leq H) = \Phi(\frac{H - mp_d}{\sqrt{p_d(1-p_d)}})$$

Analyses

- If $m = 200$, $P_k = \frac{1}{3}$ and $P_c = \frac{1}{2}$, then $P(\delta \leq \frac{m}{10} = 20) = 99.99\%$.
- If $m = 300$, $P(\delta \leq \frac{m}{10} = 20) \approx 100\%$.
- The minimal chromatic number $\chi(G)$ can be much smaller than the estimated upper bound δ

Thank you!