Investigating the Existence of Costas Latin Square via Satisfiability Testing

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Outline

Latin Square

Latin Squares

A Latin square is a $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

Costas Arrays

Costas Arrays

A Costas array of order n is a $n \times n$ array of dots and empty cells such that: (a). There are n dots and $n \times (n-1)$ empty cells, with exactly one dot in each row and column. (b). All the segments between pairs of dots differ in length or in slope.

Example

1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

Costas Latin Squares

Costas Latin Squares

A Costas Latin square of order n is a Latin square of order n such that for each symbol $i \in \{1, 2, \cdots, n\}$, a Costas array results if a dot is placed in the cells containing symbol i.

Example

1	2	4	3
2	3	1	4
3	4	2	1
4	1	3	2

Idempotency

Idempotency

For a *CLS(n)* A, we use $A(i,j)$ to denote the symbol in the i-th row and the j-th column. If A has the property that $A(i,i) = i$ for all $i \in \{1, 2, \cdots, n\}$, then it is called an idempotent Costas Latin square.

Orthogonality

Orthogonality

The orthogonality is an important property of Latin squares. For two *CLS(n)* A and B, if for all $n \times n$ positions, the pair $(A(i,j), B(i,j)), i, j \in \{1, 2, \cdots, n\}$ are different, then A and B are called orthogonal.

Example

2	3	4	1
4	1	2	3
3	2	1	4
1	4	3	2

4	3	2	1
3	4	1	2
1	2	3	4
2	1	4	3

24	33	42	11
43	14	21	32
31	22	13	44
12	41	34	23

Quasigroup

Quasigroup

A quasigroup is an algebraic structure such that the multiplication table of a finite quasigroup is a Latin square. Conversely, every Latin square can be taken as the multiplication table of a quasigroup.

Quasigroup Identities

Quasigroup Identities

The existence of quasigroups satisfying the seven short identities has been studied systematically. These identities are:

- 1. $xy \otimes yx = x$: Schröder quasigroup
- 2. $yx \otimes xy = x$: Stein’ s third law
- 3. $(xy \otimes y)y = x$: C_3-quasigroup
- 4. $x \otimes xy = yx$: Stein’ s first law; Stein quasigroup
- 5. $(yx \otimes y)y = x$
- 6. $yx \otimes y = x \otimes yx$: Stein’ s second law
- 7. $xy \otimes y = x \otimes xy$: Schröder’ s first law

Latin Squares Property

Latin Squares Property

Since in a Latin square A, each number occurs exactly once in each row and exactly once in each column, it is easy to know that:
$\forall x, y, x_1, x_2, y_1, y_2 \in N$:
$x_1 \neq x_2 \mapsto A(x_1, y) \neq A(x_2, y)$
$y_1 \neq y_2 \mapsto A(x, y_1) \neq A(x, y_2)$

Costas Property

Costas Property

For a *CLS(n) A*, the Costas property requires that for each $i \in N$, all the segments between pairs of i differ in length or in slope. This can be encoded as:

$$\forall x, y, x', y', u, v, u', v' \in N :$$
$$(A(x, y) = A(x', y') = A(u, v) = A(u', v')$$
$$\wedge (x - x' = u - u') \wedge (y - y' = v - v'))$$
$$\mapsto x = u \vee x = x'$$

Orthogonality Property

Orthogonality Property

The orthogonality property involves two *CLS(n)* *A*, *B*. This property requires that in all $n \times n$ positions, the pair $(A(i,j), B(i,j)), i, j \in N$ are different. It can be encoded as:

$\forall x_1, x_2, y_1, y_2 \in N$:

$x_1 \neq x_2 \mapsto A(x_1, y_1) \neq A(x_2, y_2) \vee B(x_1, y_1) \neq B(x_2, y_2)$

$y_1 \neq y_2 \mapsto A(x_1, y_1) \neq A(x_2, y_2) \vee B(x_1, y_1) \neq B(x_2, y_2)$

Idempotency Property

Idempotency Property

The idempotency property of *a CLS(n) A* can be encoded simply as:
$\forall x \in N : A(x, x) = x$

Quasigroup Property

Quasigroup Property

The quasigroup properties are easy to be encoded, for example, the formula for the first one is: $\forall x, y \in N$:

$\mathbf{A(}A(x,y), A(y,x)\mathbf{)} = x$

$\mathbf{A(}A(y,x), A(x,y)\mathbf{)} = x$

$\mathbf{A(}A(A(x,y),y),y\mathbf{)} = x$

$\mathbf{A(}x, A(x,y)\mathbf{)} = A(y,x)$

$\mathbf{A(}A(A(y,x),y),y\mathbf{)} = x$

$\mathbf{A(}A(y,x),y\mathbf{)} = \mathbf{A(}x, A(y,x)\mathbf{)}$

$\mathbf{A(}A(x,y),y\mathbf{)} = \mathbf{A(}x, A(x,y)\mathbf{)}$

Symmetry Breaking

Symmetry Breaking

For a *CLS(n) A*, all numbers in it are just symbols, after replacing $1, 2, \cdots, n$ by any its permutation, it is still a Costas Latin square. So the method to break symmetries for Costas Latin squares is just to fix its first column:
$\forall x \in N : A(x, 1) = x$

Transversal Matrix

Transversal

A transversal in a Latin square is a collection of positions, one from each row and one from each column, so that the elements in these positions are all different. It can be written as a vector, where the i-th element records the row index of the cell that appears in the i-th column.

Transversal Matrix

A matrix is called a transversal matrix of Latin square, if it is consisted of n mutually disjoint transversal vectors.

Construction of Transversal Matrix

Construction of Transversal

For a Latin square A of order n, we construct a matrix TA for it by this way:
If $A(i,j)=k$, then $TA(k,j)=i$, where $i, j, k \in N$.

Example

1	2	4	3
2	3	1	4
3	4	2	1
4	1	3	2

1	4	2	3
2	1	3	4
3	2	4	1
4	3	1	2

Improving for Costas Property

Improving for Costas Property

Using transversal matrix to simplify the formula for Costas property:

$$\forall x, y, z, u, v \in N :$$
$$TA(x, u) - TA(x, y) = TA(x, v) - TA(x, z) \vee u - y = v - z$$
$$\mapsto y = z \vee u = y$$

Improving for Orthogonality Property

Improving for Orthogonality Property

Using transversal matrix to reformulate the formula for orthogonality property:

$\forall x, y, u, v \in N$:

$x \neq y \mapsto TA(u, x) \neq TB(v, x) \vee TA(u, y) \neq TB(v, y)$

The Existence of Specified Properties Costas Latin Squares

Order n	Ide	Quasigroup							Ort
		.1	.2	.3	.4	.5	.6	.7	
$CLS(4)$	s	s	s	s	s	u	u	s	s
$CLS(6)$	u	u	u	u	u	u	u	u	u
$CLS(8)$	s	u	u	u	u	u	u	u	u
$CLS(10)$	u	u	u	u	u	u	u	u	*

An Idempotent Costas Latin Squares

1	3	5	7	4	2	8	6
4	2	6	8	3	1	5	7
5	7	3	1	6	8	4	2
8	6	2	4	7	5	1	3
6	8	4	2	5	7	3	1
7	5	1	3	8	6	2	4
2	4	8	6	1	3	7	5
3	1	7	5	2	4	6	8

The Run Times in Solving CLS-Ord and CLS-Ort

	SB+Tr	SB	Tr	non
CLS(6)-Ord	**0.07**	0.08	**0.07**	0.10
CLS(6)-Ort	**0.28**	2.23	TO	TO
CLS(8)-Ord	**1.39**	100.04	26.46	2207.91
CLS(8)-Ort	**67.04**	1230.96	TO	TO

The Run Times in Solving CLS-Ide and CLS-Qi

	Tr	non		Tr	non		Tr	non
CLS(6)-Ide	**0.07**	0.10	CLS(8)-Ide	**0.97**	17.79	CLS(10)-Ide	**406.59**	TO
CLS(6)-Q1	**0.07**	0.13	CLS(8)-Q1	**1.84**	36.58	CLS(10)-Q1	**351.70**	TO
CLS(6)-Q2	**0.08**	0.19	CLS(8)-Q2	**2.88**	97.45	CLS(10)-Q2	**889.16**	TO
CLS(6)-Q3	**0.07**	0.10	CLS(8)-Q3	**1.00**	2.16	CLS(10)-Q3	**12.05**	38.61
CLS(6)-Q4	**0.07**	0.09	CLS(8)-Q4	**0.98**	1.86	CLS(10)-Q4	**10.93**	36.37
CLS(6)-Q5	**0.09**	0.12	CLS(8)-Q5	**3.73**	5.83	CLS(10)-Q5	880.58	**84.13**
CLS(6)-Q6	**0.07**	0.10	CLS(8)-Q6	**0.94**	6.68	CLS(10)-Q6	**11.06**	TO
CLS(6)-Q7	**0.07**	0.10	CLS(8)-Q7	**1.01**	2.21	CLS(10)-Q7	**12.09**	TO

The Number of Clauses

	Vars	Clauses		Vars	Clauses		Vars	Clauses
CLS(6)-Odr	432	73830	CLS(8)-Odr	1024	628360	CLS(10)-Odr	2000	3245210
CLS(6)-Ide	432	73830	CLS(8)-Ide	1024	628360	CLS(10)-Ide	2000	3245210
CLS(6)-Q1-7	432	75120	CLS(8)-Q1-7	1024	622448	CLS(10)-Q1-7	2000	3255200
CLS(6)-Ort	864	186540	CLS(8)-Ort	2048	1486096	CLS(10)-Ort	4000	7390420

Thanks