Characterizing Tseitin-formulas with short regular resolution refutations

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Resolution Refutation

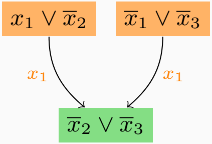
Resolution rule: $$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$ resolution on x

$x_1 \vee \overline{x}_2 \quad \overline{x}_1 \vee \overline{x}_3 \quad \overline{x}_2 \vee x_3 \quad x_2 \vee \overline{x}_3 \quad \overline{x}_1 \vee x_2 \quad x_1 \vee x_3$

Resolution Refutation

Resolution rule:

$$\frac{x_1 \lor \overline{x}_2 \qquad \overline{x}_1 \lor \overline{x}_3}{\overline{x}_2 \lor \overline{x}_3} \quad \text{resolution on } x_1$$



$\overline{x}_2 \lor x_3$ $\quad x_2 \lor \overline{x}_3$ $\quad \overline{x}_1 \lor x_2$ $\quad x_1 \lor x_3$

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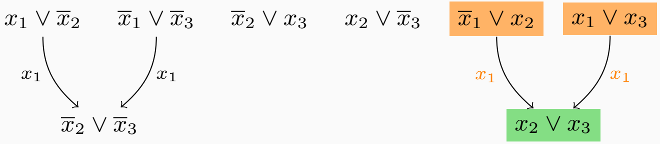
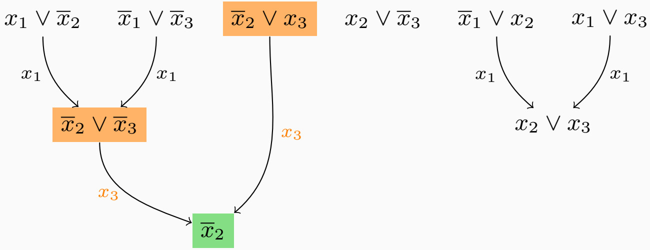


Resolution Refutation

Resolution rule: $$\frac{\overline{x}_2 \vee \overline{x}_3 \qquad \overline{x}_2 \vee x_3}{\overline{x}_2}$$ resolution on x_3



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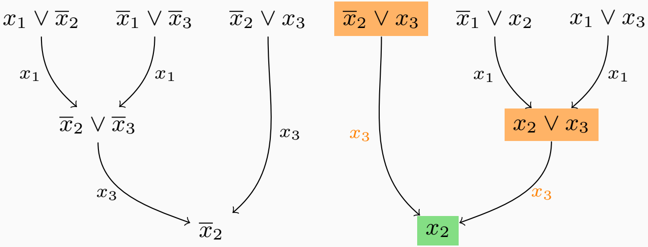
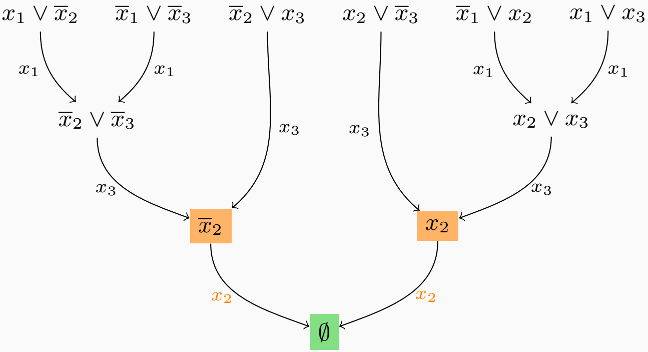


Resolution Refutation

Resolution rule: $\dfrac{\overline{x}_2 \qquad x_2}{\emptyset}$ resolution on x_2

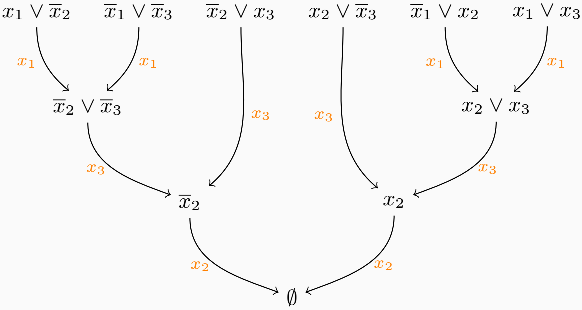


A CNF is unsat iff the clause $\emptyset$ can be derived from its clauses by resolution.

Resolution Refutation

The length of the resolution refutation is the number of the clauses in the refutation.

Regular resolution refutation of length **11**



The resolution refutation is called regular when each resolution variable occurs at most once on every path to $\emptyset$.

Resolution Refutation

Length of a refutation

Theory: each new exponential lower bound on refutations in powerful proof systems brings us closer to co-NP $\neq$ NP.

Practice: SAT solvers return refutations as proof of unsatisfiability. Long refutations mean big running times on unsat instances.

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Regular resolution refutation (RRR)

Applicable to some SAT solvers + bounds on general resolution refutation are harder show, so we assume regularity to start.

What we do

Are there unsatisfiable poly-size CNF-formulas with exponential RRR-length?

Yes, e.g.: Tseitin-formulas on expander graphs [Tseitin68, Urquhart87]

$T(G)$: an unsat Tseitin-formula for the graph G with degree bounded by a constant. Let $k = tw(G)$, $n = |var\, T(G)| = |E(G)|$.

Known already

$$2^{\Omega(\frac{k}{\log(n)})}\Omega(poly(\tfrac{1}{n})) \quad \leq \quad \text{RRR-length of } T(G) \quad \leq \quad 2^{O(k)}O(poly(n))$$

[ItsyksonRSS19] [AlekhnovichR11]

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this paper [AlekhnovichR11]

RRR-length of Tseitin-formulas (with bounded degree) are ~~almost~~ fully characterized by the treewidth.

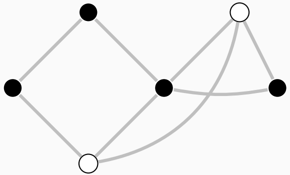
From the computational complexity blog:

"- You really want to spend your life shaving log(n) factors off ~~algorithms~~ lower bounds? - Yes I do."

Tseitin-formulas

Tseitin-formulas are CNF-formulas that are hard for many refutation systems.

$G = (V, E)$ a simple graph (undirected, no parallel edge, no self-loop) with maximum degree Δ.



Given a (black,white)-coloring of V, find a subset $\boldsymbol{E'} \subseteq E$ such that, when we keep only $\boldsymbol{E'}$,

- <u>white</u> vertices all have <u>odd</u> degree

  and

- <u>black</u> vertices all have <u>even</u> degree

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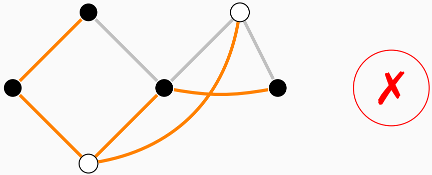

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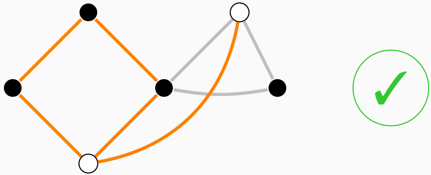

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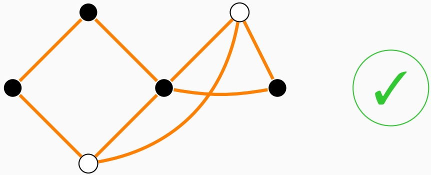

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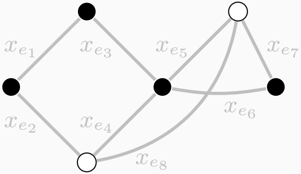
For each $e \in E$ define $x_e \in \{0,1\}$. $\boldsymbol{x_e = 1}$ iff e is in the edges kept.



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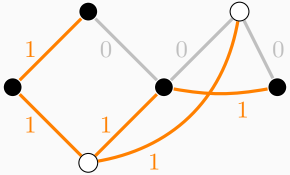

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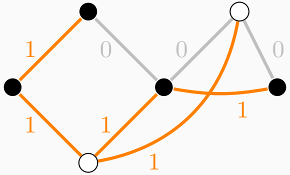

$T(G,c)$: **Tseitin-formula** for the graph G and the (white,black)-coloring c

$$T(G,c) \equiv \left(\bigwedge_{v \text{ white}} \begin{array}{l}\text{\#orange edges} \\ \text{around } v \text{ is } \underline{\text{odd}}\end{array} \right) \wedge \left(\bigwedge_{v \text{ black}} \begin{array}{l}\text{\#orange edges} \\ \text{around } v \text{ is } \underline{\text{even}}\end{array} \right)$$

#orange edges around v is odd/even = parity constraint on $x_e, e \in E(v)$

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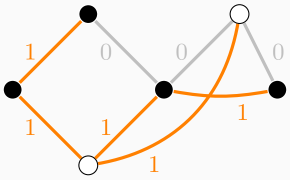

$T(G,c)$: **Tseitin-formula** for the graph G and the (white,black)-coloring c

$$T(G,c) = \bigwedge_{v \in V} F_v \equiv \left(\bigwedge_{v \text{ white}} \begin{array}{c}\text{\#orange edges} \\ \text{around } v \text{ is \underline{odd}}\end{array} \right) \wedge \left(\bigwedge_{v \text{ black}} \begin{array}{c}\text{\#orange edges} \\ \text{around } v \text{ is \underline{even}}\end{array} \right)$$

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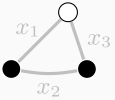
$\equiv$ CNF F_v with $\leq 2^{\Delta - 1}$ clauses.

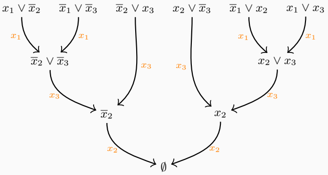
Example

$$\underbrace{(x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2)}_{x_1+x_2 \text{ is even}} \wedge \underbrace{(x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3)}_{x_2+x_3 \text{ is even}} \wedge \underbrace{(x_1 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_3)}_{x_1+x_3 \text{ is odd}}$$

is the Tseitin-formula for



It's unsat, remember that



?

$T(G, c)$ is unsat **iff** the number of white vertices in G colored by c is odd

Proof overview

Old lower bound: RRR-length of $T(G,c) \geq 2^{\Omega(\frac{k}{\log(n)})}\Omega(poly(\frac{1}{n}))$

Proof sketch:
$\left(\begin{array}{l} T(G,c) \text{ is unsat} \\ T(G,c^*) \text{ is sat} \end{array}\right)$

RRR-length of $T(G,c)$ $=$ 1-BP-size of SearchClause$(T(G,c))$

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$\geq$ $\left(\text{1-BP-size of } T(G,c^*)\right)^{\frac{1}{\log(n)}}$

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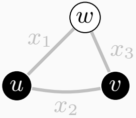
Vertex Search Problem

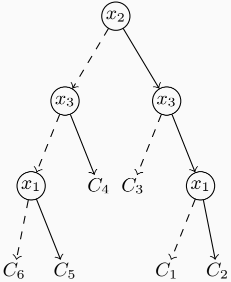
SearchVertex$(\boldsymbol{T(G,c)})$: given an assignment a, find a vertex of G whose constraint is falsified by a



u: $x_1 + x_2$ is even $\equiv C_1 \wedge C_5$

v: $x_2 + x_3$ is even $\equiv C_3 \wedge C_4$

w: $x_1 + x_3$ is odd $\equiv C_2 \wedge C_6$



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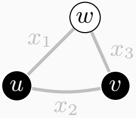

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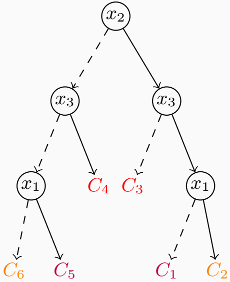

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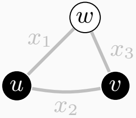

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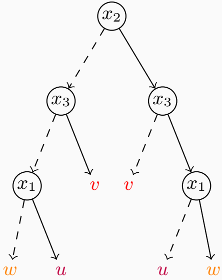

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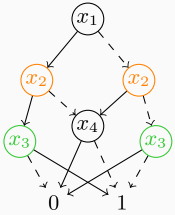
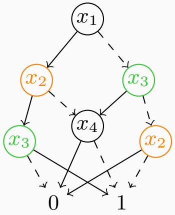
$\geq$ dec-DNNF-size of $T(G,c^*)$ (this paper)
$\geq$ $2^{\Omega(k)} \times \frac{1}{n}$ (this paper)

1-BP and DNNF

1-BP: read-once branching programs, or FBDD = OBDD with no variable order





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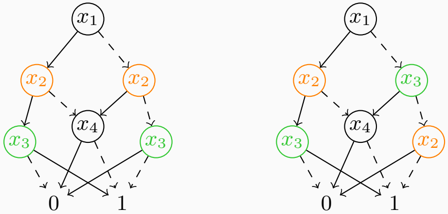

DNNF: decomposable negation normal forms
$\{\wedge, \vee\}$-circuits where the inputs of every
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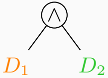

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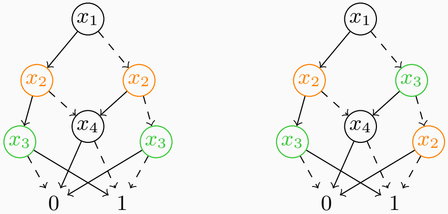

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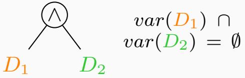
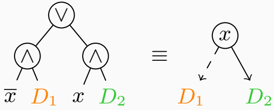


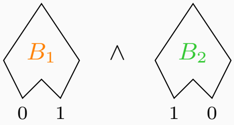
dec-DNNF: DNNF whose $\vee$-gates are of this form



The problem in the old proof

Itsykson et al. build 1-BP representing $T(G, c^*)$ satisfiable.

Problem: they sometimes need doing conjunctions of 1-BP on disjoint variables



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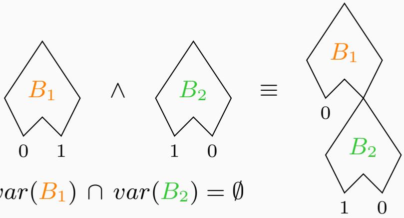

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but then B_1 is modified!

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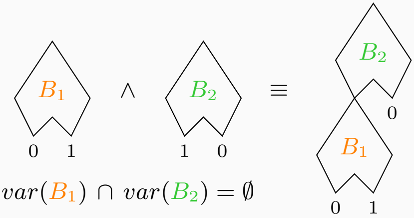

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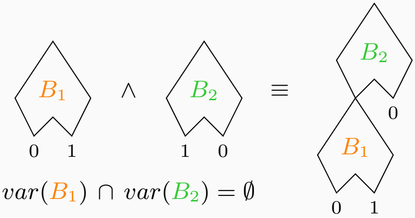
Itsykson et al. build 1-BP representing $T(G, c^*)$ satisfiable.

Problem: they sometimes need doing conjunctions of 1-BP on disjoint variables



but then B_2 is modified!

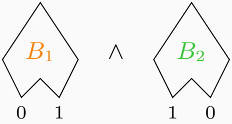
Make a copy of B_2 before, for later uses

The copies account for a $\log(n)$ exponent in the 1-BP-size of $T(G, c^*)$.

RRR-length of $T(G, c)$ $\geq$ $\left(\text{1-BP-size of } T(G, c^*)\right)^{\frac{1}{\log(n)}}$

Where our proof diverges

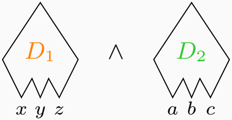
Our solution: just allow for decomposable ∧-gates in the circuit



$$var(B_1) \cap var(B_2) = \emptyset$$

Our solution: just allow for decomposable $\wedge$-gates in the circuit

- we obtain a dec-DNNF and not a 1-BP in the end

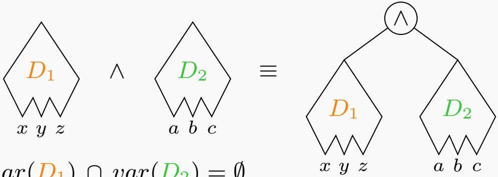


$$var(D_1) \cap var(D_2) = \emptyset$$

Where our proof diverges

Our solution: just allow for decomposable $\wedge$-gates in the circuit

- we obtain a dec-DNNF and not a 1-BP in the end
- but we never need copies

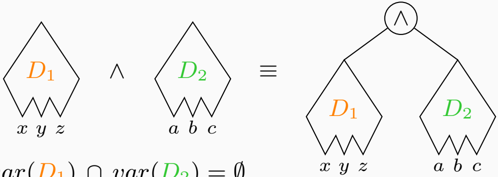


$$var(D_1) \cap var(D_2) = \emptyset$$

Where our proof diverges

Our solution: just allow for decomposable $\wedge$-gates in the circuit

- we obtain a dec-DNNF and not a 1-BP in the end
- but we never need copies



$$var(D_1) \cap var(D_2) = \emptyset$$

- so we get rid of the $\log(n)$ exponent

RRR-length of $T(G, c)$ $\geq$ dec-DNNF-size of $T(G, c^*)$

That's only half the paper!

Itsykson et al. prove

$$\text{1-BP-size of } T(G, c^*) \geq 2^{\Omega(k)}$$

we show

$$\text{DNNF-size of } T(G, c^*) \geq 2^{\Omega(k)}$$

Getting this bound requires a good understanding of Tseitin-formulas + our techniques improve on standard method for DNNF lower bounds (too technical for this presentation, see the paper).

Thank you for watching

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