XOR Local Search for Boolean Brent Equations

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- While CDCL is the dominating SAT solving paradigm, there are problems on which Stochastic Local Search performs significantly better.
  - The largest satisfiable instance of the Boolean Pythagorean Triples problem can be solved using DDFW [Divide and Distribute Fixed Weights] local search in ~one CPU minute. Other algorithms time out.
  - SLS solvers perform well in the search for new matrix multiplication schemes expressed as a SAT problem via the Boolean Brent equations.
- Can we further improve the performance of LS on a class of problems where it already performs best? We look at problems involving XOR constraints, of which matrix multiplication is one instance.

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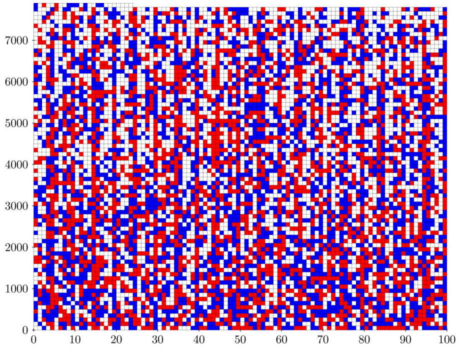
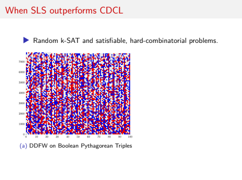


(a) DDFW on Boolean Pythagorean Triples



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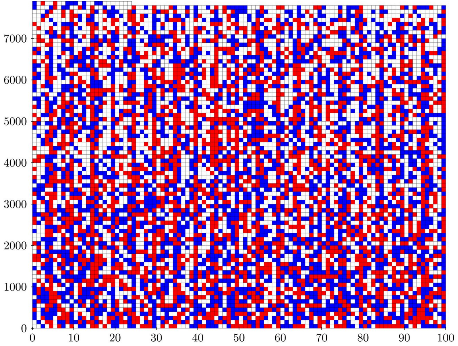
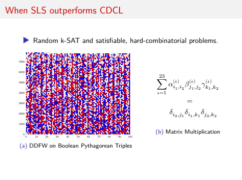


(a) DDFW on Boolean Pythagorean Triples

$$\sum_{\iota=1}^{23} \alpha^{(\iota)}_{i_1,i_2}\beta^{(\iota)}_{j_1,j_2}\gamma^{(\iota)}_{k_1,k_2} = \delta_{i_2,j_1}\delta_{i_1,k_1}\delta_{j_2,k_2}$$

(b) Matrix Multiplication



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- Most straightforwardly, we can use a direct encoding – `XOR_d`. But this produces exponentially many clauses.
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XNF

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$$(x_1 \lor x_2 \lor x_3) \land$$
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- We propose to experiment with native XOR representations more widely. In the spirit of DIMACS CNF, an *XNF* format could be used.
- Worth noting that we later found out XNF is already implemented in CryptoMiniSAT.

xnfSAT: Stochastic Local Search with native XOR

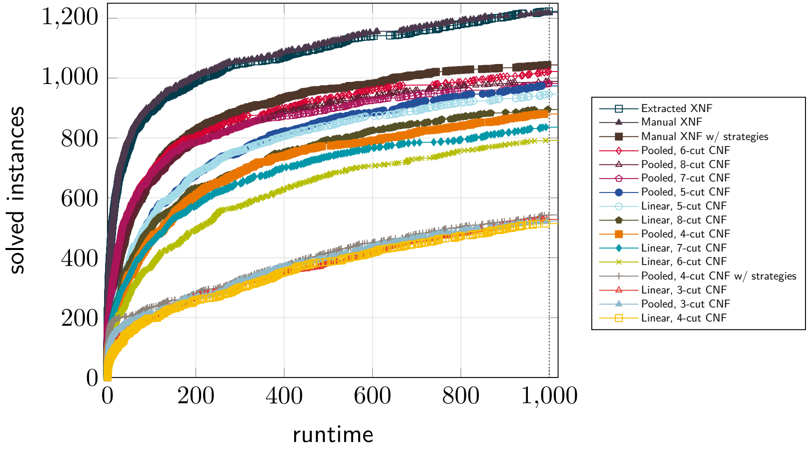
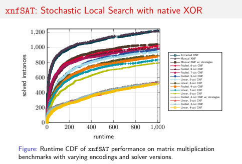


Figure: Runtime CDF of `xnfSAT` performance on matrix multiplication benchmarks with varying encodings and solver versions.



- To solve XNF, we present xnfSAT, an SLS solver supporting native XOR.
- Its performance on matrix multiplication benchmarks significantly improves upon the best CNF-based solver, YalSAT.
- To go with xnfSAT, we implemented a tool to extract XOR gates from CNF files.

We build on YalSAT

Algorithm `YalSAT`, a `WalkSAT`-based solver

for clause in input file **do**
  parse and store clause to data structure
end for
preprocess formula
$\alpha \leftarrow$ complete initial assignment of truth values
while there exists a clause falsified by α **do**
  $C \leftarrow \texttt{pickUnsatClause}()$
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- `xnfSAT` is based on `YalSAT`. Instructive to understand its outline.
- Unsurprisingly, supporting XOR needs no modifications to the high-level structure.
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Parsing and clause storage

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- Track the *parity* of each so that $\bigoplus_i x_i$ is satisfied iff $\Sigma x_i + \text{parity} \equiv 1 \mod 2$.

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- Clause and variable selection heuristics are *the most* important parts of LS solvers. `YalSAT` and consequently `xnfSAT` use `probSAT`-like selection extended with weights.
- While long OR clauses with many satisfied literals are hard to break, XOR constraints can always be broken. Length is not a good measure of the importance of an XOR constraint.

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- To make sure solving XNF strongly outperforms CNF, we performed heavy tuning on the CNF formula. We tried two variants of the Tseitin encoding.
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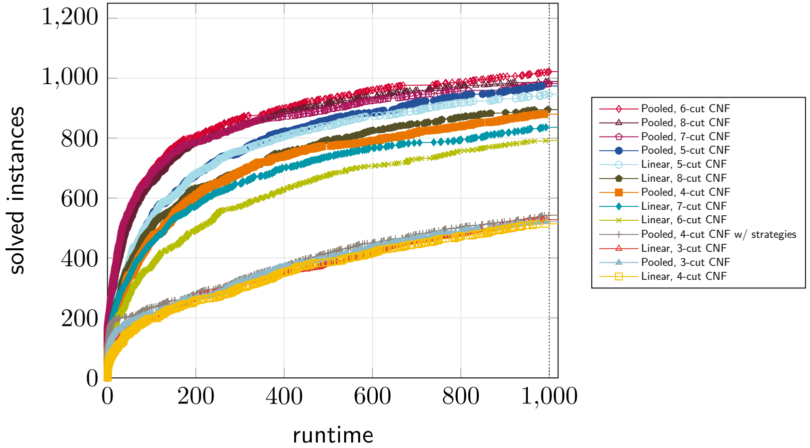
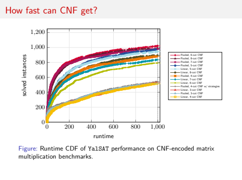


Figure: Runtime CDF of `YalSAT` performance on CNF-encoded matrix multiplication benchmarks.



- On these instances, pooled encodings are better across the board.
- Interestingly, performance initially increases with cutting number and plateaus at 6.

Not as fast as XNF!

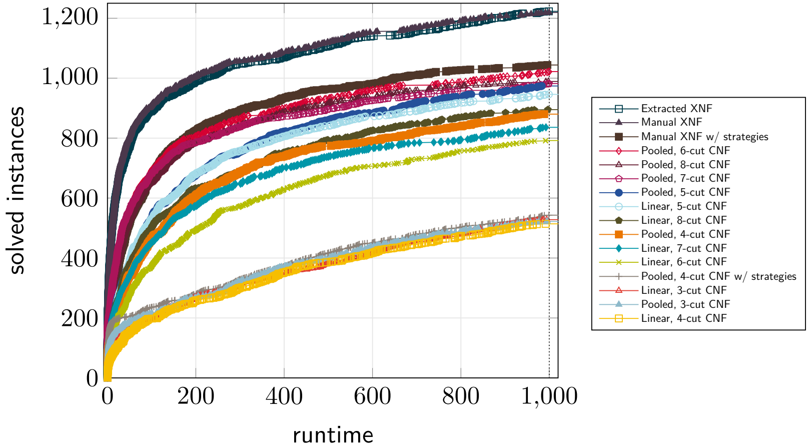
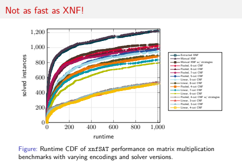


Figure: Runtime CDF of `xnfSAT` performance on matrix multiplication benchmarks with varying encodings and solver versions.



- Here, XNF solving improves over even highly tuned CNF, without having to spend any computational power on optimising the XOR encoding.
- Within a 1000s timeout, our solver operating on XNF can find between 200 and 700 more solutions compared to CNF-based runs in various configurations.

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- Observed strong performance improvements on matrix multiplication benchmarks where SLS already outperformed CDCL.
- Propose to experiment with native XOR more widely and to standardise the XNF format.
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