DQBDD: An Efficient BDD-Based DQBF Solver

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What is DQBF

- DQBF = **dependency** quantified Boolean formula
- Like QBF + **explicit** dependencies between variables
- QBF: $\forall x_1 \exists y_1 \forall x_2 \exists y_2 . (x_1 \wedge x_2) \Leftrightarrow (y_1 \Leftrightarrow y_2)$
  - y_1 depends only on x_1
  - y_2 depends on both x_1 and x_2
  - satisfiable
- DQBF: $\forall x_1 \forall x_2 \exists y_1(x_1) \exists y_2(x_2) . (x_1 \wedge x_2) \Leftrightarrow (y_1 \Leftrightarrow y_2)$
  - y_1 depends only on x_1
  - y_2 depends only on x_2
  - not satisfiable

Approach

- Main solving method: **quantifier elimination**
- Uses **binary decision diagrams** (BDD)
  - for propositional subformulas
- Enhanced by:
  - **quantifier localisation**
  - preprocessing: **HQSpre**

Quantifier Elimination

- Iteratively eliminate variables until true/false remains
- For QBF:

$$\forall x.\psi \approx \psi[1/x] \wedge \psi[0/x]$$
$$\exists y.\psi \approx \psi[1/y] \vee \psi[0/y]$$

- Same for DQBF but:
  - universal QE introduces new variables
  - existential QE is not always applicable

Universal QE

$$\forall x.\psi \approx \psi_1[1/x] \wedge \psi_2[0/x]$$

- ψ_1 is ψ but x is removed from dependency sets of existential variables
- ψ_2 is ψ_1 but each existential variable y depending on x is replaced by a new copy y'
- Example:

$$\forall \boldsymbol{x} \exists y_1(\emptyset) \exists y_2(\boldsymbol{x}).(y_1 \wedge y_2) \vee \boldsymbol{x}$$
$$\approx \exists y_1(\emptyset) \exists y_2(\emptyset) \exists y_2'(\emptyset).((y_1 \wedge y_2) \vee \mathbf{1}) \wedge ((y_1 \wedge y_2') \vee \mathbf{0})$$

Existential QE

$$\exists y(D_y).\psi \approx \psi[1/y] \vee \psi[0/y]$$

- ψ must be quantifier free
- Each variable in ψ must be either:
  - free variable
  - universal variable from D_y
  - existential variable y' with $D_{y'} \subseteq D_y$
- Example:

$$\forall x \exists y_1(\emptyset) \exists \boldsymbol{y_2}(x).(y_1 \wedge \boldsymbol{y_2}) \vee x$$
$$\approx \forall x \exists y_1(\emptyset).((y_1 \wedge \mathbf{1}) \vee x) \vee ((y_1 \wedge \mathbf{0}) \vee x)$$

Algorithm

1. Input formats
   - DQDIMACS – prenex CNF
   - DQCIR – prenex non-CNF

```
p cnf 6 2
a 1 2 0
e 3 0
a 4 0
e 5 0
d 6 1 4 0
-1 6 -3 5 0
-2 -4 5 0
```

a DQDIMACS

```
#QCIR-G14 10
forall(1, 2)
exists(3)
forall(4)
exists(5)
depend(6, 1, 4)
output(7)
8 = and(-6, 3)
9 = or(-1, -8, 5)
10 = or(-2, -4, 5)
7 = and(9, 10)
```

b DQCIR

Algorithm

1. Input formats
   - DQDIMACS – prenex CNF
   - DQCIR – prenex non-CNF
2. Apply preprocessing – HQSpre (only DQDIMACS)

```
p cnf 6 2
a 1 2 0
e 3 0
a 4 0
e 5 0
d 6 1 4 0
-1 6 -3 5 0
-2 -4 5 0
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a DQDIMACS

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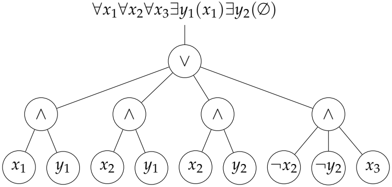
b DQCIR

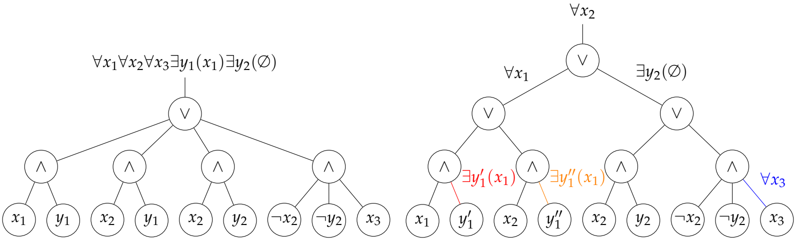
Algorithm

1. Input formats
   - DQDIMACS – prenex CNF
   - DQCIR – prenex non-CNF
2. Apply preprocessing – HQSpre (only DQDIMACS)
3. Transform into quantifier tree



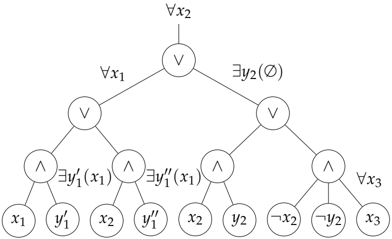
Algorithm

1. Input formats
   - DQDIMACS – prenex CNF
   - DQCIR – prenex non-CNF
2. Apply preprocessing – HQSpre (only DQDIMACS)
3. Transform into quantifier tree and push quantifiers inside



Algorithm

4. Iteratively transform to BDD while eliminating quantifiers



Algorithm

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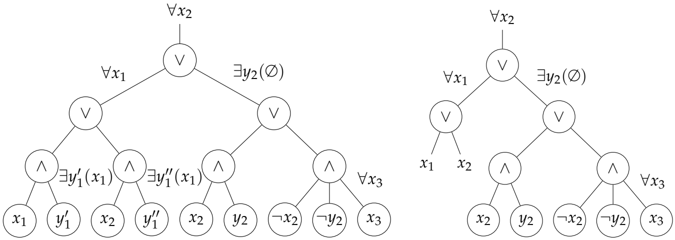

Algorithm

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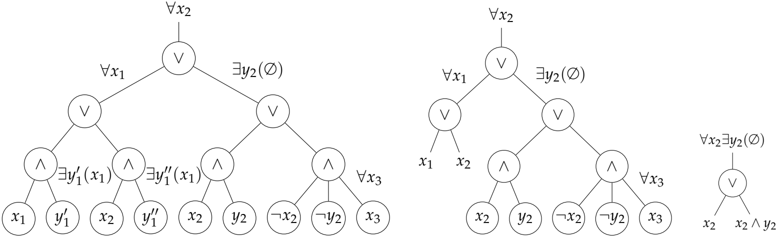
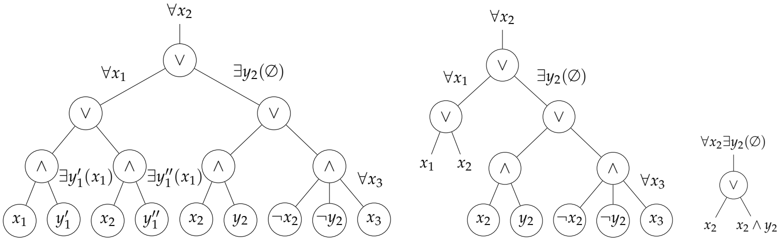


Algorithm

4. Iteratively transform to BDD while eliminating quantifiers
5. Return SAT/UNSAT if the resulting BDD represents true/false



Strategies and Heuristics

- Quantifier elimination strategies:
  - **none**
  - **simple** (default)
  - **all**
- Universal quantifier elimination order heuristics:
  - **at the beginning** (default)
  - **current lowest**
  - **vars in conjuncts**

Implementation and Usage

- Implemented in C++
- BDD library CUDD
- `https://github.com/jurajsic/DQBDD`
  - source code
  - binaries for Linux and Mac
- Demonstration

Comparison With Other Solvers

- DQBF solvers: **dCAQE**, **HQS**, **iDQ**, **iProver**
- Preprocessing (**HQSpre**) used for all solvers
- Benchmarks:
  - 3277 **partial equivalence checking** instances } next slides
  - 404 **controller synthesis problem** instances } next slides
  - 22 **SAT** instances encoded as DQBF – worse than others
- **Winner** of the DQBF track of QBF Evaluation 2020

Comparison With Other Solvers – PEC

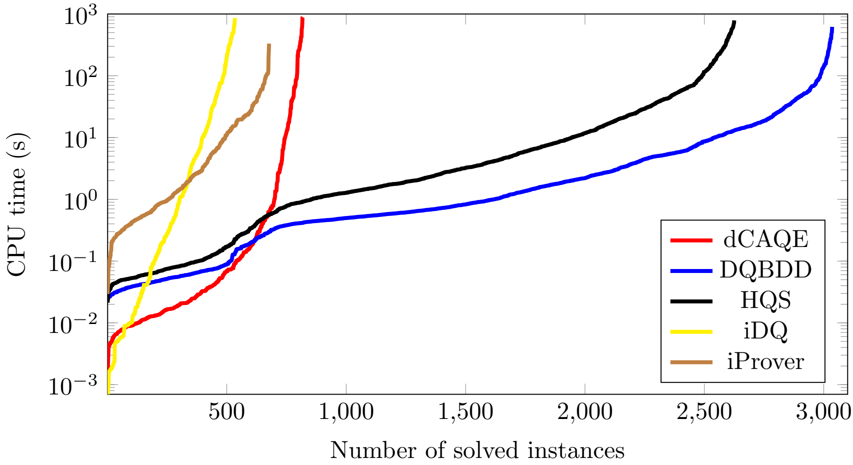


Figure: Cactus plot comparing DQBF solvers for instances of partial equivalence checking

Comparison With Other Solvers – CSP

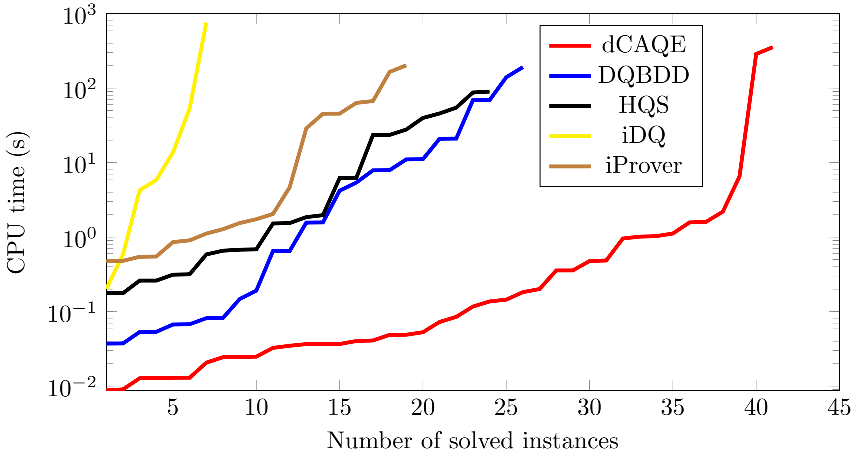


Figure: Cactus plot comparing DQBF solvers for instances of controller synthesis problem

Thank You for Your Attention!