Girard quantales, their linear orders, and completely distributive lattices*

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In the first part of this talk I'll define linear orders valued in a Girard quantale, a variant of the usual notion of metric space whose distance is valued in a quantale. I'll give examples of these structures mostly arising from combinatorics and geometry. In particular, if the quantale $Q_{\vee}([0,1])$ of sup-preserving endofunctions of the unit interval is considered, linear orders on a set of finite cardinality n valued in it can be identified with images of continuous monotone paths in the n-dimensional cube $[0,1]^n$, linking the origin to the unit vector [3].

Considering that the quantale of sup-preserving endofunctions of a complete lattice is Girard if and only if the lattice is completely distributive (and that complete chains are completely distributive), the linear orders described above have motivated further research on completely distributive lattices, see e.g. [2]. In a second part of this talk I'll focus on constructing completely distributive lattices from given ones, using complete congruences. Indeed, it is an elementary observation that the quotient of a completely distributive lattice by a complete congruence is again completely distributive. In a recent work [1] we have given a geometrical characterisation (as sublocales) of these complete congruences. The characterisation relies on Hoffmann-Lawson duality between completely distributive lattices and their spectra. I'll illustrate the characterisation with the posets $(0,1]$ and $[0,1)^{op} \times (0,1]$, that are the spectra of $[0,1]$ and $Q_{\vee}([0,1])$, respectively. In particular, I'll argue that complete congruences give rise to a frame, and that such frame is not, in general, Boolean, nor completely distributive.

[1] C. Calk and L. Santocanale. Complete congruences of completely distributive lattices. Submitted to the conference RAMiCS 2024, Mar. 2024.

[2] C. de Lacroix and L. Santocanale. Frobenius structures in star-autonomous categories. In B. Klin and E. Pimentel, editors, *CSL 2023*, volume 252 of *LIPIcs*, pages 18:1–18:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023.

[3] M. J. Gouveia and L. Santocanale. The continuous weak order. *Journal of Pure and Applied Algebra*, 225(2):106472, 2021.

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